Problem Related to Maxima and Minima of Function of two variables

1. Maximum: →

* The Function $f(x,y)$ is said to have a Maximum value at a point (a,b) if
$f(x,y) - f(a,b) < 0$ for every point (x,y) in the nhd of (a,b)

2. Minima → The Function $f(x,y)$ is said to have a minimum value at a point (a,b) provided $f(x,y) - f(a,b) > 0$ for every Point (x,y) in the nhd of (a,b)

* Necessary Condition under which $f(x,y)$ to have Extreme value at (a,b) are

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0 \qquad \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0$$

* Stationary point → the point (x,y) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ is called stationary point

* Sufficient Condition → usually we denote $\frac{\partial^2 f}{\partial x^2} = A$ or r

$\frac{\partial^2 f}{\partial x \partial y} = B$ or s

and $\frac{\partial^2 f}{\partial y^2} = C$ or t

$f(x,y)$ is (i) Maximum at (a,b) if $r < 0$ and $rt - s^2 > 0$

(ii) Minimum at (a,b) if $r > 0$ and $rt - s^2 > 0$

(iii) Neither Maximum nor Minimum at (a,b) if $rt - s^2 < 0$ $r \neq 0$

Working Rule for Finding Maxima and Minima of a function $f(x,y)$

STEP I → Calculate

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y} \text{ and } \frac{\partial^2 f}{\partial y^2}$$

STEP II → Solve the Simultaneous equation

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$

STEP III – Calculate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at the point which are obtained for solving Simultaneous Equation $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

STEP → if $r < 0$ and $rt - s^2 > 0$ at (a,b)
$\Rightarrow f(x,y)$ have maximum value at (a,b)

if $r > 0$ and $rt - s^2 > 0$ at (a,b)
$\Rightarrow f(x,y)$ will be minimum at (a,b)

if $rt - s^2 < 0$ then $f(x,y)$ have neither maximum nor minimum.

Problem

* Find the maximum and minimum value of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 10$

Here $f(x,y) = x^3 + y^3 - 3x - 12y + 10$

$f_x = 3x^2 - 3$ $\qquad f_y = 3y^2 - 12$

$r = f_{xx} = 6x$ $\qquad t = f_{yy} = 6y$

$s = f_{xy} = 0$

Ex:→ Find the maximum or minimum values of $f(x,y)$ where

$$f(x,y) = x^3 + y^3 - 3axy \quad x \neq 0,\ y \neq 0$$

Sol^n → Here given function

$$f(x,y) = x^3 + y^3 - 3axy$$

Now $f_x(x,y) = 3x^2 - 3ay$ $\qquad f_{xx}(x,y) = 6x = r$

$f_y(x,y) = 3y^2 - 3ax$ $\qquad f_{xy}(x,y) = -3a = s$

$\qquad f_{yy}(x,y) = 6y = t$

for maximum or minimum we calculate stationary points

for which $f_x = 0$ — (I) and $f_y = 0$ — (II)

$\Rightarrow 3x^2 - 3ay = 0$ $\qquad 3y^2 - 3ax = 0$

$\Rightarrow x^2 = ay$ $\qquad \Rightarrow y^2 = ax$

$\Rightarrow x^4 = a^2y^2$ $\qquad$ Since $y^2 = ax$

$= a^2 \cdot ax$

$\Rightarrow \cancel{a^3}\, x^4 - a^3 x = 0$

$x(x^3 - a^3) = 0$ $\qquad$ When $x = a$

$\Rightarrow x = 0$ or $x = a$ $\qquad y^2 = a^2$

$\qquad \Rightarrow y = a$

Here stationary point is (a,a)

Now $\cancel{rt - s^2}$ $\qquad f_{xx}(a,a) = 6a = r$

$f_{xy}(a,a) = -3a = s$

$f_{yy}(a,a) = 6a = t$

Now $rt - s^2 = 36a^2 + 9a^2 = 45a^2 > 0$

Here $r = 6a$ $\qquad rt - s^2 > 0$

When $a = +ve$, $r > 0$ and $rt - s^2 > 0 \Rightarrow$ it has minimum value at (a,a)

When $a = -ve \Rightarrow r < 0$ and $rt - s^2 > 0 \Rightarrow$ it has maximum value at (a,a)

for Stationary points

$f_x = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$f_y = 0 \Rightarrow 3y^2 - 12 = 0 \Rightarrow y^2 - 4 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

Stationary points $(1, 2)$, $(1, -2)$, $(-1, 2)$, $(-1, -2)$

Since $r = f_{xx} = 6x$

$s = f_{xy} = 0$

$t = f_{yy} = 6y$

(1) At the point $(1,2)$ $\quad r = 6$
$s = 0$
$t = 12$

Now $rt - s^2$
$= 72 > 0$ $\quad$ Since $r = 6 > 0$

$\therefore$ $f(x,y)$ has Minimum value at $(1,2)$

and Since $f(x,y) = x^3 + y^3 - 3x - 12y + 10$

$\therefore$ $f(1,2) = 1 + 8 - 3 - 24 + 10$

Here $r > 0$ and $(rt - s^2 > 0 \Rightarrow f(x,y)$ has minimum.
value at $(1,2)$ is -8

(ii) When Stationary point is $(1, -2)$

$r = 6x = 6 > 0$ $\quad$ Now $r > 0$

$s = 0$

$t = -12$ $\quad$ Also $rt - s^2 = -72 < 0$

Thus $f(x,y)$ has Neither Maximum nor Minimum.

(iii) When Stationary Point $(-1, 2)$

$r = 6x = -6$ $\quad$ Now $rt - s^2 = -72 < 0$

$s = 0$ $\quad$ and $r < 0$

$t = 6y = 12$ $\Rightarrow$ $f(x,y)$ has Neither Maximum nor Minimum.

(iv) When Stationary Point $(-1, -2)$ $\quad r = -6$, $s = 0$, $t = -12$

$r < 0$ $\quad rt - s^2 > 0$

$\Rightarrow f(x,y)$ has Maximum Point at $(-1, -2)$

Topic
B.Sem. Lagrange's Method of undetermined Multipliers

Usually Lagrange's method of undetermined multipliers can only determine points where the function can have a maximum or minimum. It can not say definitely whether a given point is a maximum or minimum. This is also true in the case of three variables

Problem 1. Find the minimum value of $x^2+y^2+z^2$ having given $ax+by+cz=p$

Solution Let $u = x^2+y^2+z^2 \Rightarrow du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$\Rightarrow du = 2x\,dx + 2y\,dy + 2z\,dz.$

for stationary points

$du = 0$

$\Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 0$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$ ——— (1)

Also say $\phi(x,y,z) = ax+by+cz-p = 0$

$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$= a\,dx + b\,dy + c\,dz = 0$ ——— (2)

Now $du + \lambda d\phi = 0$

$\Rightarrow x\,dx + y\,dy + z\,dz + \lambda[a\,dx + b\,dy + c\,dz] = 0$

$\Rightarrow (x+a\lambda)dx + (y+b\lambda)dy + (z+\lambda c)dz = 0$

$\Rightarrow x + a\lambda = 0$ ——— (3)

$y + b\lambda = 0$ ——— (4)

$z + c\lambda = 0$ ——— (5)

value for λ (3) $\times x$ + (4) $\times y$ + (5) $\times z$

We have $x^2 + ax\lambda + y^2 + by\lambda + z^2 + cz\lambda = 0$

$\Rightarrow x^2+y^2+z^2 + \lambda(ax+by+cz) = 0$

$\Rightarrow u + \lambda p = 0$ ——— (6)

Again $a\times 3 + b\times (4) + c\times 5$

$\Rightarrow ax + a^2\lambda + by + b^2\lambda + cz + c^2\lambda = 0$

$\Rightarrow (ax+by+cz) + \lambda(a^2+b^2+c^2) = 0$

$\Rightarrow p + (a^2+b^2+c^2)\lambda = 0$ —(7) Since, by the question $ax+by+cz=p$

Now from (6) $u + \lambda p = 0$

$$\Rightarrow \lambda = -\frac{u}{p}$$

From (7) $(a^2+b^2+c^2)\lambda + p = 0$

$$\Rightarrow (a^2+b^2+c^2)\lambda = -p$$

$$\Rightarrow \lambda = \frac{-p}{a^2+b^2+c^2}$$

$$\therefore \quad -\frac{u}{p} = \frac{-p}{a^2+b^2+c^2}$$

$$\Rightarrow u = \frac{p^2}{a^2+b^2+c^2}$$

<u>Now</u> we determine the minimum value of u

Differentiating

$$ax+by+cz=p$$

<u>Partially</u> with respect to x. Treating z as a function of x and y.

We get $a + c\frac{\partial z}{\partial x} = 0 \Rightarrow c\frac{\partial z}{\partial x} = -a$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{a}{c}$$

Also differentiating $ax+by+cz=p$ with respect to y Treating z as a function of x and y

$$b + c\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{b}{c}$$

Now Given $u = x^2+y^2+z^2$

$$\frac{\partial u}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} = 2x + 2z\left(-\frac{a}{c}\right) = 2x - \frac{2az}{c}$$

$$\frac{\partial u}{\partial y} = 2y + 2z\frac{\partial z}{\partial y} = 2y + 2z\left(-\frac{b}{c}\right) = 2y - \frac{2bz}{c}$$

$$r = \frac{\partial^2 u}{\partial x^2} = 2 - \frac{2a}{c}\frac{\partial z}{\partial x} = 2 - \frac{2a}{c}\left(-\frac{a}{c}\right) = 2 + \frac{2a^2}{c^2}$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -\frac{2a}{c}\frac{\partial z}{\partial y} = -\frac{2a}{c}\left(-\frac{b}{c}\right) = \frac{2ab}{c^2}$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2 - \frac{2b}{c}\frac{\partial z}{\partial y} = 2 - \frac{2b}{c}\left(-\frac{b}{c}\right) = 2 + \frac{2b^2}{c^2}$$

Here $r>0$

Also $rt-s^2 = \left(2+\frac{2a^2}{c^2}\right)\left(2+\frac{2b^2}{c^2}\right) - \frac{4a^2b^2}{c^4}$

$= 4 + \frac{4b^2}{c^2} + \frac{4a^2}{c^2} + \frac{4b^2a^2}{c^4} - \frac{4a^2b^2}{c^4}$

$= 4\left(1+\frac{b^2}{c^2}+\frac{a^2}{c^2}\right) > 0$

Thus we get minimum value of u as above

Find the maximum and minimum values of $x^2+y^2+z^2$ subject to conditions

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$$

and $z = x+y$

Solution :→ Here objective function $f(x,y,z) = x^2+y^2+z^2$

Consider Lagrangian function F of independent variables x, y, z

$$F = x^2+y^2+z^2 + \lambda_1\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right) + \lambda_2(x+y-z) \quad \text{—— (1)}$$

$\therefore dF = 2x\,dx + 2y\,dy + 2z\,dz + \lambda_1\left(\frac{2x}{4}dx + \frac{2y}{5}dy + \frac{2z}{25}dz\right) + \lambda_2(dx+dy-dz)$

$= \left(2x+\frac{x}{2}\lambda_1+\lambda_2\right)dx + \left(2y+\frac{2y}{5}\lambda_1+\lambda_2\right)dy + \left(2z+\frac{2z}{25}\lambda_1-\lambda_2\right)dz$

Since x, y, z are independent variables, we get

$2x + \frac{x}{2}\lambda_1 + \lambda_2 = 0 \qquad \therefore x = -\frac{2\lambda_2}{\lambda_1+4}$

$2y + \frac{2y}{5}\lambda_1 + \lambda_2 = 0 \qquad y = \frac{-5\lambda_2}{2\lambda_1+10}$

$2z + \frac{2z}{25}\lambda_1 - \lambda_2 = 0 \qquad z = \frac{25\lambda_2}{2\lambda_1+50}$

Since by the question

$x+y = z$

$-\frac{2\lambda_2}{\lambda_1+4} - \frac{5\lambda_2}{2\lambda_1+10} = \frac{25\lambda_2}{2\lambda_1+50}$

$\Rightarrow \frac{2}{\lambda_1+4} + \frac{5}{2\lambda_1+10} + \frac{25}{2\lambda_1+50} = 0$ —— $\lambda_2 \neq 0$ (1)

λ if $\lambda_2 = 0$

$x = y = z = 0$ but $(0,0,0)$ does not satisfy the other condition given in question.

Now from (1)

$$\frac{2}{\lambda_1+4} + \frac{5}{2\lambda_1+10} = -\frac{25}{2\lambda_1+50}$$

$$\frac{4\lambda_1+20+5\lambda_1+20}{2\lambda_1^2+18\lambda_1+40} = -\frac{25}{2\lambda_1+50}$$

$\Rightarrow (9\lambda_1+40)(2\lambda_1+50) = -25(2\lambda_1^2+18\lambda_1+40)$

$\Rightarrow 18\lambda_1^2+450\lambda_1+80\lambda_1+2000 = -50\lambda_1^2-450\lambda_1-1000$

$\Rightarrow 68\lambda_1^2+900\lambda_1+80\lambda+3000 = 0$

$\Rightarrow 68\lambda_1^2+980\lambda_1+3000 = 0$

$17\lambda_1^2+245\lambda_1+750 = 0$

this is a quadratic equation in λ_1

so that $\lambda_1 = -10$

$= -\frac{75}{17}$

for $\lambda_1 = -10$ $\quad x = \frac{1}{3}\lambda_2$

$y = \frac{1}{2}\lambda_2$

$z = 5/6\,\lambda_2$

Putting the value of x, y, z in

$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ On solving

we get $\lambda_2^2 = \frac{180}{19} \Rightarrow \lambda_2 = \pm 6\sqrt{\frac{5}{19}}$

$\therefore$ corresponding stationary points are

$\pm 2\sqrt{5/19}, \pm 3\sqrt{\frac{5}{19}}, \pm 5\sqrt{\frac{5}{19}}$

$x^2+y^2+z^2 = \frac{20}{19} + \frac{45}{19} + \frac{125}{19} = \frac{190}{19} = 10$

When $\lambda_1 = -\frac{75}{17}$

$x = \frac{34}{7}\lambda_2$

$y = -\frac{17}{4}\lambda_2$

$z = \frac{17}{28}\lambda_2$

putting these value of x, y, z
in

$$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$$

we get $\lambda_2 = \frac{\pm 140}{17\sqrt{646}}$

$\therefore$ Corresponding stationary points.

$$\left(\pm \frac{40}{\sqrt{646}}, \mp \frac{35}{\sqrt{646}}, \pm \frac{5}{\sqrt{646}} \right)$$

and the value of $x^2+y^2+z^2$ corresponding to these point is $\frac{75}{17}$

$\therefore$ Maximum value $= 10$
Minimum value $= \frac{75}{17}$